On Products arising from infinitely many factors*

Leonhard Euler

§1 If in Analysis one gets to quantities of such a kind which can neither be expressed by rational nor irrational numbers, usually infinite expressions are used to define these quantities; these infinite expressions are to be considered the more appropriate for this purpose the faster one gets to cognition and estimation of the quantities expressed by them, if they are used. Therefore, expressions of this kind are used very often to represent the values of transcendental quantities, for example, logarithms, circular arcs and other quantities defined by the quadrature of curves; and applying those expressions we get to an exact cognition both of logarithms and circular arcs and even of other transcendental quantities. Yes, infinite expressions of this kind even have an extraordinary use to define irrational quantities and roots of algebraic equations approximately; they, if their use is considered, in most cases are to be preferred to the true expressions by far.

§2 But several species, very different to each other, of the infinite expressions of this kind are to be constituted; the first of these species contains all infinite series, consisting of infinitely many terms affected with the signs $+$ and $-$; this doctrine is now certainly developed so far that not only many methods to express so algebraic as transcendental quantities of this kind by infinite series exist, but also, having propounded an infinite series, methods have

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§3 It is convenient to refer those expressions, which consist of innumerable factors, to another other species of infinite expressions; although many expressions of this kind have already been found and are known, nevertheless still neither a way to obtain them nor to discover their values was ever explained. But the expressions of this species seem to be equally worth to be developed as the first consisting of infinite series and considering them will be useful for the whole field of Analysis. For, furthermore, since expressions of this kind show the nature of the quantities they describe very clearly and are often more than appropriate to find approximate values, they have a tremendous use to form the logarithms of the quantities in consideration, which is very often immensely helpful for calculations. So, if any arbitrary quantity X was transformed into an expression of this kind

$$\frac{a}{\alpha}\cdot\frac{b}{\beta}\cdot\frac{c}{\gamma}\cdot\frac{d}{\delta}\cdot\frac{e}{\varepsilon}\cdot\text{etc.},$$

one will immediately have the logarithm of the quantity X,

$$\ln\left(\frac{a}{\alpha}\right)+\ln\left(\frac{b}{\beta}\right)+\ln\left(\frac{c}{\gamma}\right)+\ln\left(\frac{d}{\delta}\right)+\text{etc}$$

which series converges the more, the closer those factors get to 1. Therefore, I decided to start the development of the theory of infinite expressions of this kind and share my observations on them in this dissertation, that it is easier for others to expand it further some time.

§4 The first who published an expression containing infinitely many factors of this kind was Wallis in his book *Arithmetica infinitorum*, where he showed, if the diameter of the circle is $=1$ that the area of the circle will be

$$\frac{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdot 10\cdot 10\cdot 12}{3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdot 9\cdot 11\cdot 11}\text{etc.};$$

he deduced this expression from the interpolation of this series

$$\frac{2}{3}+\frac{2\cdot 4}{3\cdot 5}+\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}+\text{etc.},$$

whose intermediate terms he demonstrated to depend on the quadrature of the circle.

Therefore, since these expressions must have their origin in the interpolation of series, it seemed to be appropriate to begin this treatise on infinite products with interpolations. For, after in the fifth tome of our COMMENTARII[1] I had given a method to find interpolations using quadratures of curves, it was known at the same time, transcendental quantities of what kind infinite products resulting this way exhibit.

§5 Therefore, I consider the following progression

$$\overset{1}{(f+g)}+\overset{2}{(f+g)(f+2g)}+\overset{3}{(f+g)(f+2g)(f+3g)}+\overset{4}{(f+g)(f+2g)(f+3g)(f+4g)}+\text{etc.}$$

whose arbitrary term corresponding to the index n, is found from the preceding by multiplying it by $f+ng$; but I showed in the mentioned dissertation that the term corresponding to the index n of this series is

$$=\frac{g^{n+1}\int dx(-\ln(x))^n}{(f+(n+1)g)\int x^{f:g}dx(1-x)^n}$$

having done each of both integrations in such a way that the integrals vanish having put $x=0$ and then having set $x=1$ after the integration. Therefore, this expression will at the same time indicate, on which quadrature the single intermediate terms depend. For, although, if n is a fractional number, it is not obvious, which quadrature the quantity $\int dx(-\ln x)^n$ expresses, I nevertheless also showed that, having put $\frac{p}{q}$ instead of n, the formula $\int dx(-\ln x)^{\frac{p}{q}}$ is equal to

$$\sqrt[q]{1\cdot 2\cdot 3\cdots p\left(\frac{2p}{q}+1\right)\left(\frac{3p}{q}+1\right)\left(\frac{4p}{q}+1\right)\cdots\left(\frac{qp}{q}+1\right)}$$
$$\times\int dx(x^2-x^3)^{\frac{p}{q}}\cdot\int dx(x^3-x^4)^{\frac{p}{q}}\cdot\int dx(x^4-x^5)^{\frac{p}{q}}\cdots\int dx(x^{q-1}-x^q)^{\frac{p}{q}};$$

using this reduction the value of $\int dx(-\ln(x))^{\frac{p}{q}}$ can be expressed by means of algebraic curves.

[1]Euler refers to his paper "De progressionibus transcendentibus seu quarum termini generales algebraice dari nequeunt". This is E19 in the Eneström-Index.

§6 If now in the assumed series the term corresponding to the index $\frac{1}{2}$ is put z, from the law of the series the terms corresponding to the indices are $\frac{3}{2}, \frac{5}{2}, \frac{7}{2}$ etc. will be the following:

$$z + z\left(f + \frac{3}{2}g\right) + z\left(f + \frac{3}{2}g\right)\left(f + \frac{5}{2}g\right) + z\left(f + \frac{3}{2}g\right)\left(f + \frac{5}{2}g\right)\left(f + \frac{7}{2}g\right) + \text{etc.}$$

But since the assumed progression is finally confounded with the geometric progression, these interpolated terms will finally become the geometric means of two contiguous terms of the series. Hence, if the single interpolated terms are considered as the geometric means from the beginning, the following approximations to the term corresponding to the index z will result.

$$\begin{array}{lll} \text{I.} & z = & \sqrt{f+g} \\ \text{II.} & z = & \sqrt{\dfrac{(f+g)(f+g)(f+2g)}{1\left(f+\frac{3}{2}g\right)\left(f+\frac{3}{2}g\right)}} \\ \text{III.} & z = & \sqrt{\dfrac{(f+g)(f+g)(f+2g)(f+2g)(f+3g)}{1\left(f+\frac{3}{2}g\right)\left(f+\frac{3}{2}g\right)\left(f+\frac{5}{2}g\right)\left(f+\frac{5}{2}g\right)}} \\ & \text{etc.} & \end{array}$$

Using this law of progression it is understood that the term corresponding to the index $\frac{1}{2}$ approximately is

$$= (f+g)^{\frac{1}{2}}\sqrt{\frac{(f+g)(f+2g)(f+2g)(f+3g)(f+3g)(f+4g)(f+4g)(f+5g)(f+5g)(f+6g)}{\left(f+\frac{3}{2}g\right)\left(f+\frac{3}{2}g\right)\left(f+\frac{5}{2}g\right)\left(f+\frac{5}{2}g\right)\left(f+\frac{7}{2}g\right)\left(f+\frac{7}{2}g\right)\left(f+\frac{9}{2}g\right)\left(f+\frac{9}{2}g\right)\left(f+\frac{11}{2}g\right)\left(f+\frac{11}{2}g\right)}\text{etc.}}$$

§7 Therefore, now it is not only certain that the term of the assumed series

$$\overset{1}{(f+g)} + \overset{2}{(f+g)(f+2g)} + \overset{3}{(f+g)(f+2g)(f+3g)} + \text{etc.}$$

corresponding to the index $\frac{1}{2}$ is exhibited by this infinite expression, but also that the same expression is reduced to quadratures of curves. For, having put $n = \frac{1}{2}$, because of $p = 1$ and $q = 2$, it is

$$\int dx(-\ln(x))^{\frac{1}{2}} = \sqrt{1 \cdot 2 \int dx\sqrt{x - xx}};$$

this expression integrated in the corrected way gives the square root of the area of the circle, whose diameter is $= 1$; or having put the ratio of the diameter to the circumference $1 : \pi$ it will be

$$\int dx(-\ln(x))^{\frac{1}{2}} = \frac{\sqrt{\pi}}{2}$$

Therefore, the same term corresponding to the index $\frac{1}{2}$ and we put z is found to be

$$= \frac{h\sqrt{\pi g}}{(2f+3g)\int x^{f:g}dx\sqrt{1-x}} = \frac{\sqrt{\pi g}}{(2f+3g)\int y^{f+g-1}dy\sqrt{1-y^g}}$$

having treated the integral in the same way as it was prescribed before. But by means of the reduction of integral formulas of this kind it is

$$\int y^{f+g-1}dy\sqrt{1-y^g} = \frac{2fg}{(2f+g)(2f+3g)}\int \frac{y^{f-1}dy}{\sqrt{1-y^g}} = \frac{2f}{2f+3g}\int y^{f-1}dy\sqrt{1-y^g}$$

Having substituted these expressions one finds

$$\frac{(2f+g)(2f+3g)(2f+3g)(2f+5g)(3f+5g)(2f+7g)}{(2f+2g)(2f+2g)(2f+4g)(2f+4g)(2f+4g)(2f+6g)} \text{ etc.}$$

$$= \frac{2ff(2f+g)}{\pi g}\left(\int y^{f-1}dy\sqrt{1-y^g}\right)^2 = \frac{2ffg}{\pi(2f+g)}\left(\frac{y^{f-1}dy}{\sqrt{1-y^g}}\right)^2$$

Therefore, by means of this equation innumerable quadratures can be transformed into infinite products and vice versa the value of infinite products of this kind can be transformed in quadratures of curves.

§8 To illustrate this equality by examples, let $g = 1$ and it will be

$$\int y^{f-1}dy\sqrt{1-y} = \frac{2\cdot 2\cdot 4\cdot 6\cdot 8\cdot 10\cdots(2f-2)}{3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13\cdots(2f-1)} \text{ etc.}$$

Hence it will be

$$\frac{2ff(2f+1)\cdot 2\cdot 2\cdot 2\cdot 2\cdot 4\cdot 4\cdots(2f-2)(2f-2)}{\pi\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdots(2f+1)(2f+1)} \text{ etc.} = \frac{(2f+1)(2f+3)(2f+3)}{(2f+2)(2f+2)(2f+4)}\text{etc.}$$

which expression ordered or reduced to continuity gives

$$\pi = 4 \cdot \frac{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot 10 \cdot 10}{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot 9 \cdot 11} \text{ etc.},$$

which is the Wallis formula itself and results, whatever positive integer number is substituted for f. This same expression results, if one puts $g = 2$ and $f =$ an arbitrary odd integer.

§9 Therefore, because it is

$$\frac{fg}{\pi}\left(\int \frac{y^{f-1}dy}{\sqrt{1-y^g}}\right)^2 = \frac{(2f+g)(2f+g)(2f+3g)(2f+3g)(2f+5g)(2f+5g)}{2f(2f+2g)(2f+2g)(2f+4g)(2f+3g)(2f+6g)} \text{ etc.},$$

in the same way it will be

$$\frac{hk}{\pi}\left(\frac{y^{h-1}dy}{\sqrt{1-y^k}}\right)^2 = \frac{(2h+k)(2h+k)(2h+3k)(2h+3k)(2h+5k)(2h+5k)}{2h(2h+2k)(2h+2k)(2h+4k)(2h+4k)(2h+6k)} \text{ etc.}$$

Hence having divided this expression by the first one will obtain the following equation not containing circumference of the circle π

$$\frac{dg(\int y^{f-1}dy : \sqrt{1-y^g})^2}{hk(\int y^{h-1}dy : \sqrt{1-y^k})^2} = \frac{2h(2f+g)^2(2h+2k)^2(2f+3g)^2(2h+4k)^2(2f+5g)^2}{2f(2h+k)^2(2f+2g)^2(2f+3h)^2(2f+4g)^2(2h+5k)^2} \text{ etc.};$$

this equation, having extracted the square root, yields this equation

$$\frac{\int y^{f-1}dy : \sqrt{1-y^g}}{\int y^{h-1}dy : \sqrt{1-y^k}} \cdot \sqrt{\frac{g}{k}} = \frac{2h(2f+g)(2f+2k)(2f+3g)(2h+4k)(2f+5g)}{2f(2h+k)(2f+2g)(2h+3k)(2f+4g)(2h+5k)} \text{ etc.}$$

§10 But this infinite expression does not have a constant value; for, even though it is continued to infinity, it nevertheless has one value, if an even number of factors is taken, but another value, if an odd number is taken. Therefore, if it is not $k = g$, in which case is does not matter, at which point the multiplication is interrupted, two factors must be combined; having done so one will obtain two equations, depending on whether an even or an odd number of factors was taken. But first having expanded the general expression accurately one will obtain

$$\frac{g\int y^{f-1}dy : \sqrt{1-y^g}}{k\int y^{h-1}dy : \sqrt{1-y^k}}$$

$$= \frac{2h(2f+g)}{2f(2h+k)} \cdot \frac{(2f+2g)(2h+3k)}{(2h+2k)(2f+3g)} \cdot \frac{(2f+4g)(2h+5k)}{(2h+4k)(2f+5g)} \cdot \frac{(2f+6g)(2h+7k)}{(2h+6k)(2f+7g)} \cdot \text{etc.}$$

But by taking the other pairs of terms it will be

$$\frac{f\int y^{f-1}dy : \sqrt{1-y^g}}{h\int y^{h-1}dy : \sqrt{1-y^k}}$$

$$= \frac{(2f+g)(2h+2k)}{(2h+k)(2f+2g)} \cdot \frac{(2f+3g)(2h+4k)}{(2h+3k)(2f+4g)} \cdot \frac{(2f+5g)(2h+6k)}{(2h+5k)(2f+6g)} \cdot \frac{(2f+7g)(2h+8k)}{(2h+7k)(2f+8g)} \cdot \text{etc.}$$

in which expressions the spots, where it is possible to interrupt the operation, are marked by points.

§11 But let us consider the case with more attention, in which it is $k = g$; for, in this case the infinite expression can certainly be considered to consist of simple factors, and it will be

$$\frac{\int y^{f-1}dy : \sqrt{1-y^g}}{\int y^{h-1}dy : \sqrt{1-y^g}} = \frac{2h(2f+g)(2h+2g)(2f+3g)(2h+4g)}{2f(2h+g)(2f+2g)(2h+3g)(2f+4g)} \text{ etc.;}$$

that this expression is less confounded with the preceding one because of the same letters, let us put $2f = a$ and $2h = b$ and $y = x^2$ here; having substituted these letters it will result

$$\frac{\int x^{a-1}dx : \sqrt{1-x^{2g}}}{\int x^{b-1}dx : \sqrt{1-x^{2g}}} = \frac{b(a+g)(b+2g)(a+3g)(b+4g)(a+5g)}{a(b+g)(a+2g)(b+3g)(a+4g)(b+5g)} \text{ etc.,}$$

which expression compared to the first given in § 9, which having equally put $y = x^2$ goes over into this one

$$\frac{4fg}{\pi}\left(\frac{x^{2f-1}dx}{\sqrt{1-x^{2g}}}\right)^2 = \frac{(2f+g)(2f+g)(2f+3g)(2f+3g)(2f+5g)(2f+5g)}{2f(2f+2g)(2f+2g)(2f+4g)(2f+4g)(2f+6g)} \text{ etc.,}$$

will reveal extraordinary properties, whose truth can otherwise hardly be demonstrated.

§12 For, it is immediately plain, if one puts $a=3f$, $b=2f+g$, that the last infinite expression is transformed into this one; therefore, also the expressions equal to those and containing the quadratures of curves will become equal to each other in this case, whence the following equality follows

$$\frac{\int x^{2f-1}dx:\sqrt{1-y^{2g}}}{\int x^{2f+g-1}dx:\sqrt{1-x^{2g}}}=\frac{4fg}{\pi}\left(\int x^{2f-1}dx:\sqrt{1-x^{2g}}\right)^2,$$

if after the integration one puts $x=1$, of course. Therefore, hence it follows that it will be

$$\pi=4fg\int\frac{x^{2f-1}dx}{\sqrt{1-x^{2g}}}\cdot\int\frac{x^{2f+g-1}dx}{\sqrt{1-x^{2g}}};$$

or having put $2f=a$ it will be

$$\pi=2ag\int\frac{x^{a-1}dx}{\sqrt{1-x^{2g}}}\cdot\int\frac{x^{a+g-1}dx}{\sqrt{1-x^{2g}}};$$

this certainly is a very remarkable theorem, since using it one can assign the product of two integral formulas, of which often none can be exhibited.

§13 The truth of this theorem is certainly easily checked in the cases, in which the one integral formula either admits an integration or depends on the quadrature of the circle. For, let us put $g=1$ and $a=1$; of course, it will be

$$\pi=2\int\frac{dx}{\sqrt{1-x^2}}\cdot\int\frac{xdx}{\sqrt{1-x^2}},$$

for,

$$2\int\frac{dx}{\sqrt{1-x^2}}$$

having put $x=1$ after the integration gives the quantity π itself and

$$\int\frac{xdx}{\sqrt{1-xx}}=1-\sqrt{1-xx}$$

becomes $=1$ having put $x=1$. In like manner, if it is $a=2$ while it still is $g=1$, it is understood that it will be

$$\pi=4\int\frac{xdx}{\sqrt{1-xx}}\cdot\int\frac{xxdx}{\sqrt{1-xx}};$$

for, it is

$$\int \frac{xdx}{\sqrt{1-xx}} = 1 \quad \text{and} \quad \int \frac{xxdx}{\sqrt{1-xx}} = \frac{\pi}{4};$$

in these cases the truth of the theorem is confirmed from elsewhere.

§14 But the remaining cases, in which none of both integral formulas can exhibited either explicitly or by means of the quadrature of the circle, yield as many curious and remarkable theorems. So, having put $g = 2$ and $a = 1$ it will be

$$\pi = 4\int \frac{dx}{\sqrt{1-x^4}} \cdot \int \frac{xxdx}{\sqrt{1-x^4}},$$

where

$$\int \frac{xxdx}{\sqrt{1-x^4}}$$

exhibits the ordinate in the *curva elastica rectangula*[2],

$$\int \frac{dx}{\sqrt{1-x^4}}$$

on the other hand expresses the arc corresponding to the abscissa x of the *elastica*. Therefore, the rectangle of the arc corresponding to the abscissa 1 of the *elastica* and the corresponding ordinate will become equal to the area of the circle, whose diameter is that abscissa 1; this property of the *elastica* can maybe hardly or not even be seen and demonstrated at all by another method.

§15 But before I leave this case of the *elastica*, it will be helpful to have expressed the integral by means of an ordinary series at least in the case, in which it is $x = 1$. For, since it is

$$\frac{1}{\sqrt{1-x^4}} = \frac{(1+x^2)^{-\frac{1}{2}}}{\sqrt{1-x^2}}$$

and

$$(1+xx)^{-\frac{1}{2}} = 1 - \frac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}x^4 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \text{etc.},$$

[2]rectangular elastic curve

the single terms will depend on the quadrature of the circle. But having done both integrations for the case $x = 1$ it will be

$$\int \frac{dx}{\sqrt{1-x^4}} = \frac{\pi}{2}\left(1 - \frac{1}{4} + \frac{1\cdot 9}{4\cdot 16} - \frac{1\cdot 9\cdot 25}{4\cdot 16\cdot 36} + \text{etc}\right)$$

and

$$\int \frac{x^2}{\sqrt{1-x^4}} = \frac{\pi}{2}\left(\frac{1}{2} - \frac{1\cdot 3}{4\cdot 4} + \frac{1\cdot 9\cdot 5}{4\cdot 16\cdot 6} - \frac{1\cdot 9\cdot 25\cdot 7}{4\cdot 16\cdot 36\cdot 8} + \text{etc}\right)$$

But hence by approximation it results

$$\int \frac{dx}{\sqrt{1-x^4}} = \frac{5}{6}\cdot\frac{\pi}{2} \quad \text{and} \quad \int \frac{xxdx}{\sqrt{1-x^4}} = \frac{3}{5}.$$

§16 If it was $a = 1$, it will be

$$\pi = 2g\int \frac{dx}{\sqrt{1-x^{2g}}}\cdot\int \frac{x^g dx}{\sqrt{1-x^{2g}}}$$

which two integral expressions are of such a nature that, if

$$\int \frac{x^g dx}{\sqrt{1-x^{2g}}}$$

is the ordinate corresponding to the curve of a certain abscissa x, that

$$\int \frac{dx}{\sqrt{1-x^{2g}}}$$

will be the length of the same curve. Therefore, if in this curve the abscissa is taken as $x = 1$, the product or the rectangle of the ordinate by the length of the curve will have the same ratio to the area of the circle, whose diameter is the abscissa $x = 1$, as 2 to the number g; this proposition holds, as long as g was a positive number; for, negative values are excluded automatically.

§17 If $a - 1$ is assumed smaller than g so that a and g are prime to each other, one will have the following remarkable theorems; for, if it is

$$a + g - 1 > 2g,$$

then the integration can be reduced to a simpler formula.

$$\begin{array}{l|l}
\pi = 2\int \frac{dx}{\sqrt{1-x^2}} \cdot \int \frac{xdx}{\sqrt{1-x^2}} & \pi = 24\int \frac{x^2dx}{\sqrt{1-x^8}} \cdot \int \frac{x^6dx}{\sqrt{1-x^8}} \\
\pi = 4\int \frac{dx}{\sqrt{1-x^4}} \cdot \int \frac{x^2dx}{\sqrt{1-x^4}} & \pi = 10\int \frac{dx}{\sqrt{1-x^{10}}} \cdot \int \frac{x^5dx}{\sqrt{1-x^{10}}} \\
\pi = 6\int \frac{dx}{\sqrt{1-x^6}} \cdot \int \frac{x^3dx}{\sqrt{1-x^6}} & \pi = 20\int \frac{xdx}{\sqrt{1-x^{10}}} \cdot \int \frac{x^6dx}{\sqrt{1-x^{10}}} \\
\pi = 12\int \frac{xdx}{\sqrt{1-x^6}} \cdot \int \frac{x^4dx}{\sqrt{1-x^6}} & \pi = 30\int \frac{x^2dx}{\sqrt{1-x^{10}}} \cdot \int \frac{x^7dx}{\sqrt{1-x^{10}}} \\
\pi = 8\int \frac{dx}{\sqrt{1-x^8}} \cdot \int \frac{x^4dx}{\sqrt{1-x^8}} & \pi = 40\int \frac{x^3dx}{\sqrt{1-x^{10}}} \cdot \int \frac{x^8dx}{\sqrt{1-x^{10}}} \\
\pi = 12\int \frac{dx}{\sqrt{1-x^{12}}} \cdot \int \frac{x^6dx}{\sqrt{1-x^{12}}} & \pi = 28\int \frac{xdx}{\sqrt{1-x^{14}}} \cdot \int \frac{x^8dx}{\sqrt{1-x^{14}}} \\
\pi = 60\int \frac{x^4dx}{\sqrt{1-x^{12}}} \cdot \int \frac{x^{10}dx}{\sqrt{1-x^{12}}} & \pi = 42\int \frac{x^2dx}{\sqrt{1-x^{14}}} \cdot \int \frac{x^9dx}{\sqrt{1-x^{14}}} \\
\pi = 14\int \frac{dx}{\sqrt{1-x^{14}}} \cdot \int \frac{x^7dx}{\sqrt{1-x^{14}}} & \pi = 56\int \frac{x^3dx}{\sqrt{1-x^{14}}} \cdot \int \frac{x^{10}dx}{\sqrt{1-x^{14}}}
\end{array}$$

$$\pi = 70\int \frac{x^4dx}{\sqrt{1-x^{14}}} \cdot \int \frac{x^{11}dx}{\sqrt{1-x^{14}}}$$

§18 Therefore, having found all this the reduction of integral formulas to simpler ones is also significantly promoted. For, since these two integral formulas

$$\int \frac{x^m dx}{\sqrt{1-x^{2g}}} \quad \text{and} \quad \int \frac{x^{m+n}}{\sqrt{1-x^{2g}}}$$

could have still only be reduced to each other, if n was a multiple of the exponent $2g$, so this reduction now also succeeds, if n was only a multiple of g, in the case $x = 1$, of course. But as, if n is a product of the exponent g by an even number, the quotient resulting from the division of the one formula by the other, is easily assigned, so vice versa, if n is the product of g by an odd number, then the product of these formulas will be assigned very easily.

§19 Therefore, all these things reduce to this, that, if the integral of this formula was known

$$\int \frac{x^m dx}{\sqrt{1-x^{2g}}}$$

in the case, in which it is $x = 1$, that in the same case also the integral of this formula

$$\int \frac{x^{m+n}dx}{\sqrt{1-x^{2g}}},$$

can be exhibited, if n is a multiple of g. For, let A be the integral of the formula

$$\int \frac{x^{m}dx}{\sqrt{1-x^{2g}}}$$

in the case, in which it is $x = 1$; the integral of the other formula by putting g, $2g$, $3g$ etc. successively for n will have the following values

$$\begin{aligned}
\int \frac{x^{m}dx}{\sqrt{1-x^{2g}}} &= A \\
\int \frac{x^{m+g}dx}{\sqrt{1-x^{2g}}} &= \frac{\pi}{2(m+1)gA} \\
\int \frac{x^{m+2g}dx}{\sqrt{1-x^{2g}}} &= \frac{(m+1)A}{m+g+1} \\
\int \frac{x^{m+3g}dx}{\sqrt{1-x^{2g}}} &= \frac{(m+g+1)\pi}{2(m+1)(m+2g+1)gA} \\
\int \frac{x^{m+4g}dx}{\sqrt{1-x^{2g}}} &= \frac{(m+1)(m+2g+1)A}{(m+g+1)(m+3g+1)} \\
\int \frac{x^{m+5g}dx}{\sqrt{1-x^{2g}}} &= \frac{(m+g+1)(m+3g+1)\pi}{2(m+1)(m+2g+1)(m+4g+1)gA} \\
&\text{etc.}
\end{aligned}$$

§20 Further, since this general integral formula

$$\int x^{m+ig}dx(1+x^2)^{k-\frac{1}{2}}$$

while i and k denote arbitrary integer numbers can be reduced to this formula

$$\int \frac{x^{m+ig}dx}{\sqrt{1-x^{2g}}},$$

it is understood that the integral of this very far-extending formula $\int x^{m+ig}dx(1-x^{2g})^{k-\frac{1}{2}}$ can be assigned using the integral

$$\int \frac{x^{m}dx}{\sqrt{1-x^{2g}}},$$

known at least in the case, in which it is $x = 1$ after the integration. But the cases, in which i is an odd number, except for this integral, also require the quadrature of the circle π.

§21 Therefore, as I was led to the nature of these integral formulas by the term corresponding to the index $\frac{1}{2}$ of the series assumed above in § 5, so it will be worth one's while to investigate the other intermediate terms in like manner. Therefore, let the term corresponding to the index $\frac{p}{q}$ be in question, and let us put that term $= z$; hence the following terms will be:

$$\begin{array}{ccccc} \frac{p}{q} & & \frac{p+q}{q} & & \frac{p+2q}{q} \\ z+ & & \frac{z(fq+(p+q))}{q} & + & \frac{z(fq+(p+q)g)(fq+(p+2q)g)}{q^2} + \text{etc.} \end{array}$$

Now, by considering in the same way that this progression finally goes over into the geometric one, the following series of approximations to the term z will result:

$$\text{I.} \quad z = 1(f+g)^{\frac{p}{q}}$$

$$\text{II.} \quad \frac{z(fq+(p+q)g}{q} = (f+g)^{\frac{q-p}{q}}(f+g)^{\frac{p}{q}}(f+2g)^{\frac{p}{q}}$$

$$\text{III.} \quad z\left(f+\frac{p+q}{q}g\right)\left(f+\frac{p+2q}{q}g\right) = (f+g)^{\frac{p-q}{q}}(f+g)^{\frac{p}{q}}(f+2g)^{\frac{q-p}{q}}(f+2g)^{\frac{p}{q}}(f+3g)^{\frac{p}{q}}$$

therefore, hence the true value of z will be found

$$\frac{(f+g)^{\frac{p}{q}}(f+g)^{\frac{q-p}{q}}(f+2g)^{\frac{p}{q}}(f+2g)^{\frac{q-p}{q}}(f+3g)^{\frac{p}{q}}(f+2g)^{\frac{q-p}{q}}}{1\cdot\left(f+\frac{p+q}{q}g\right)^{\frac{p}{q}}\left(f+\frac{p+q}{q}g\right)^{\frac{q-p}{q}}\left(f+\frac{p+2q}{q}g\right)^{\frac{p}{q}}\left(f+\frac{p+2q}{q}g\right)^{\frac{q-p}{q}}\left(f+\frac{p+3q}{q}g\right)^{\frac{p}{q}}} \text{ etc.}$$

Or having made some small changes so that the infinitesimal factors become $= 1$ and the expression can be interrupted at any arbitrary place, it will be

$$\frac{z}{\left(f+\frac{p}{q}g\right)^{\frac{p}{q}}} = \frac{(f+g)^{\frac{p}{q}}}{\left(f+\frac{p}{q}g\right)^{\frac{p}{q}}}\cdot\frac{(f+g)^{\frac{q-p}{q}}}{\left(f+\frac{p+q}{q}g\right)^{\frac{q-p}{q}}}\cdot\frac{(f+2g)^{\frac{p}{q}}}{\left(f+\frac{p+q}{q}g\right)^{\frac{p}{q}}}$$

$$\cdot\frac{(f+2g)^{\frac{q-p}{q}}}{\left(f+\frac{p+2q}{q}g\right)^{\frac{q-p}{q}}}\cdot\frac{(f+3g)^{\frac{p}{q}}}{\left(f+\frac{p+2q}{q}g\right)^{\frac{p}{q}}}\cdot\text{etc.};$$

the structure how the factors proceed is immediately clear.

§22 But the value of the same intermediate term z can be expressed by means of the general term of this series

$$z=\frac{g^{\frac{p+q}{q}}\int dx(-\ln(x))^{\frac{p}{q}}}{\left(f+\frac{p+q}{q}g\right)\int x^{f:g}dx(1-x)^{\frac{p}{q}}}.$$

Hence, if one puts

$$\int dx(-\ln(x))^{\frac{p}{q}}=\sqrt[q]{1\cdot 2\cdot 3\cdots p\left(\frac{2p}{q}+1\right)\left(\frac{3p}{q}+1\right)\left(\frac{4p}{q}+1\right)\cdots\left(\frac{qp}{q}+1\right)}$$

$$\times\int dx(x-x^2)^{\frac{p}{q}}\cdot\int dx(x^2-x^3)^{\frac{p}{q}}\cdot\int dx(x^3-x^4)^{\frac{p}{q}}\cdots\int dx(x^{q-1}-x^q)^{\frac{p}{q}}=\sqrt[q]{P}$$

and $x=y^g$, in which case it is

$$\int x^{f:g}dx(1-x)^{\frac{p}{q}}=g\int y^{f+g-1}dx(1-y^g)^{\frac{p}{q}},$$

$$=\frac{ggp}{fq+(p+q)g}\int\frac{y^{f+g-1}dy}{(1-y^g)^{\frac{q-p}{q}}}=\frac{pfqq}{q\left(f+\frac{p}{q}g\right)\left(f+\frac{p+q}{q}g\right)}\int\frac{y^{f-1}dy}{(1-y^g)^{\frac{q-p}{q}}}$$

and further it is put

$$\int\frac{y^{f-1}dy}{(1-y^g)^{\frac{q-p}{q}}}=Q,$$

it will be

$$z=\frac{q\left(f+\frac{p}{q}g\right)p^{\frac{1}{q}}}{pfg^{\frac{q-p}{q}}Q}$$

§23 Now, having substituted the infinite expression for z and having taken the power of the exponent g, this equation will result

$$\frac{q^qP}{p^qf^pg^{q-p}Q^q}=\frac{f^{q-p}}{\left(f+\frac{p}{q}g\right)^{q-p}}\cdot\frac{(f+g)^p}{\left(f+\frac{p}{q}g\right)^p}\cdot\frac{(f+g)^{q-p}}{\left(f+\frac{p+q}{q}\right)^{q-p}}\cdot\frac{(f+2g)^p}{\left(f+\frac{p+q}{q}g\right)^p}\cdot\frac{(f+2g)^{q-p}}{\left(f+\frac{p+2q}{q}g\right)^{q-p}}\cdot\text{etc.}$$

Therefore, if in like manner it is put

$$\int\frac{y^{h-1}dy}{(1-y^g)^{\frac{q-p}{q}}}=R,$$

it will be

$$\frac{p^q h^p g^{q-p} R^q}{q^q P} = \frac{\left(h+\frac{p}{q}g\right)^{q-p}}{h^{q-p}} \cdot \frac{\left(h+\frac{p}{q}g\right)^{p}}{(h+g)^p} \cdot \frac{\left(h+\frac{p+q}{q}g\right)^{q-p}}{(h+g)^{q-p}} \cdot \text{etc.},$$

which two expressions multiplied by each other will give

$$\frac{h^p R^q}{f^p Q^q} = \frac{f^{q-p}(h+\frac{p}{q}g)(f+g)^q(h+\frac{p+q}{q}g)^q(f+2g)^q(h+\frac{p+sq}{q})^q}{h^{q-p}(f+\frac{p}{q}g)(h+g)^q(f+\frac{p+q}{q}g)^q(h+2g)^q(f+\frac{p+sq}{q})^q} \text{ etc.}$$

§24 Therefore, if both sides are multiplied by $\frac{f^p}{h^p}$ and the root of the power q is taken, one will find

$$\frac{R}{Q} = \frac{f\left(h+\frac{p}{q}g\right)(f+g)\left(h+\frac{p+q}{q}g\right)(f+2g)\left(h+\frac{p+2q}{q}g\right)}{h\left(f+\frac{p}{q}g\right)(h+g)\left(f+\frac{p+q}{q}g\right)(h+2g)\left(f+\frac{p+2q}{q}g\right)} \text{ etc}$$

$$= \frac{\int y^{h-1}dy(1-y^g)^{\frac{p-q}{q}}}{\int y^{f-1}dy(1-y^g)^{\frac{p-q}{q}}};$$

in these integrals, since they were taken in such a way that they vanish having put $y=0$, it must be $y=1$; having done so one will find the value of the propounded infinite expression exhibited by means of quadratures. Therefore, by means of this infinite expression the one quadrature can be reduced to the other, if one puts $y=1$, of course.

§25 But that we are hence led to comparisons of integral formulas of this kind as in the first case, in which it was $p=1$ and $q=2$, let us put $p=1$ and $q=3$ here and it will be

$$P = \frac{10}{3}\int dx(x-x^2)^{\frac{1}{3}} \cdot \int dx(x^2-x^3)^{\frac{1}{3}}$$

and

$$Q = \int \frac{y^{h-1}dy}{(1-y^g)^{\frac{2}{3}}},$$

Therefore, it will be

$$\frac{27P}{fg^2Q^3} = \frac{ff(f+g)(f+g)(f+g)(f+2g)}{(f+\frac{1}{3}g)(f+\frac{1}{3}g)(f+\frac{1}{3}g)(f+\frac{4}{3}g)(f+\frac{4}{3}g)(f+\frac{4}{3}g)} \text{ etc}$$

and

$$\frac{R}{Q} = \frac{f\left(h+\frac{1}{3}g\right)(f+g)\left(h+\frac{4}{3}g\right)(f+2g)\left(h+\frac{7}{3}g\right)}{h\left(f+\frac{1}{3}g\right)(h+g)\left(f+\frac{4}{3}g\right)(h+2g)\left(f+\frac{7}{3}g\right)} \text{ etc.,}$$

which two expressions, since in that one revolution consists of three factors, but here of two factors, cannot be transformed into each other, whatever is substituted for h.

§26 Therefore, let it be

$$S = \int \frac{y^{k-1}dy}{(1-y^g)^{\frac{2}{3}}};$$

it will be

$$\frac{S}{Q} = \frac{f\left(k+\frac{1}{3}g\right)(f+g)\left(k+\frac{4}{3}g\right)(f+2g)\left(k+\frac{7}{3}g\right)}{k\left(f+\frac{1}{3}g\right)(k+g)\left(f+\frac{4}{3}g\right)(k+2g)\left(f+\frac{7}{3}g\right)} \text{ etc.,}$$

which expression combined with the preceding will give

$$\frac{RS}{Q^2} = \frac{ff\left(h+\frac{1}{3}g\right)\left(k+\frac{1}{3}g\right)(f+g)(f+g)\left(h+\frac{4}{3}g\right)}{hk\left(f+\frac{1}{3}g\right)\left(f+\frac{1}{3}g\right)(h+g)(k+g)\left(f+\frac{4}{3}g\right)} \text{ etc.,}$$

which expression will be converted into that one equal to $\frac{27P}{fg^2Q^3}$ by putting

$$h = f + \frac{1}{3}g \quad \text{and} \quad k = f + \frac{2}{3}g.$$

Therefore, one will have this equation

$$\frac{27P}{fg^2} = QRS$$

or having substituted the true values it will be

$$90\int dx(x-x^2)^{\frac{1}{3}} \cdot \int dx(x^2-x^3)^{\frac{1}{3}} = fg^2 \int \frac{y^{f-1}dy}{(1-y^g)^{\frac{2}{3}}} \cdot \int \frac{y^{f+\frac{1}{3}g-1}dy}{(1-y^g)^{\frac{2}{3}}} \cdot \int \frac{y^{f+\frac{2}{3}g-1}dy}{(1-y^g)^{\frac{2}{3}}}.$$

§27 But before we prosecute this any further, it will be convenient to attribute a more beautiful form to the value of P in general. But because having put $x = z^q$ it is

$$\int dx(x^n - x^{n+1})^{\frac{p}{q}} = \frac{npq}{(n+1)((n+1)p+q)} \int \frac{z^{np-1}dz}{(1-z^q)^{\frac{q-p}{q}}},$$

after the substitution this equation will result

$$P = 1 \cdot 2 \cdot 3 \cdots p \cdot \frac{p^{q-1}}{q} \int \frac{z^{p-1}dz}{(1-z^p)^{\frac{q-p}{q}}} \cdot \int \frac{z^{3p-1}dz}{(1-z^q)^{\frac{q-p}{q}}} \cdots \int \frac{z^{(q-1)p-1}dz}{(1-z^q)^{\frac{q-p}{q}}}.$$

If the root of the power q of this expression is pulled, the value of $\int dx(-\ln x)^{\frac{p}{q}}$ will result.

§28 Now having put $p = 1$ and $q = 3$ it will be

$$P = \frac{1}{3} \int \frac{dz}{(1-z^3)^{\frac{2}{3}}} \cdot \int \frac{zdz}{(1-z^3)^{\frac{2}{3}}}.$$

But having put $y = z^3$ one will obtain the following equation

$$\int \frac{dz}{(1-z^3)^{\frac{2}{3}}} \cdot \int \frac{zdz}{(1-z^3)^{\frac{2}{3}}} = 3fg^2 \int \frac{z^{3f-1}dz}{(1-z^{3g})^{\frac{2}{3}}} \cdot \int \frac{z^{3f+g-1}dz}{(1-z^{3g})^{\frac{2}{3}}} \cdot \int \frac{z^{3f+2g-1}dz}{(1-z^{3g})^{\frac{2}{3}}}.$$

If one now puts $3f = a$, the following remarkable equation will result

$$\int \frac{dz}{(1-z^3)^{\frac{2}{3}}} \int \frac{zdz}{(1-z^3)^{\frac{2}{3}}} = ag^2 \int \frac{z^{a-1}dz}{(1-z^{3g})^{\frac{2}{3}}} \cdot \int \frac{z^{a+g-1}dz}{(1-z^{3g})^{\frac{2}{3}}} \int \frac{z^{a+2g-1}dz}{(1-z^{3g})^{\frac{2}{3}}}.$$

This one compared to the one above

$$\int \frac{dz}{\sqrt{1-z^2}} = ag \int \frac{z^{a-1}dz}{\sqrt{1-z^{2g}}} \cdot \int \frac{z^{a+g-1}dz}{\sqrt{1-z^{2g}}}$$

already indicates clearly enough, how the following equations of this kind will look.

§29 But before I risk it to conclude anything by induction, I want to actually expand some cases. Therefore, let $p=2$ and $q=3$ and hence it will be found

$$P=\frac{8}{3}\int\frac{zdz}{(1-z^3)^{\frac{1}{3}}}\cdot\int\frac{z^3dz}{(1-z^3)^{\frac{1}{3}}}=\frac{8}{9}\int\frac{dz}{(1-z^3)^{\frac{1}{3}}}\cdot\int\frac{zdz}{(1-z^3)^{\frac{1}{3}}},$$

$$Q=\int\frac{y^{f-1}dy}{(1-y^g)^{\frac{1}{3}}},\quad R=\int\frac{y^{h-1}dy}{(1-y^g)^{\frac{1}{3}}}.$$

But the infinite expression will be then:

$$\frac{27P}{8f^2gQ^3}=\frac{f(f+g)(f+g)(f+g)(f+2g)(f+2g)}{\left(f+\frac{2}{3}g\right)\left(f+\frac{2}{3}g\right)\left(f+\frac{2}{3}g\right)\left(f+\frac{5}{3}g\right)\left(f+\frac{5}{3}g\right)\left(f+\frac{5}{3}g\right)}\text{ etc.}$$

and

$$\frac{R}{Q}=\frac{f\left(h+\frac{2}{3}g\right)(f+g)\left(h+\frac{5}{3}g\right)(f+2g)\left(h+\frac{8}{3}g\right)}{h\left(f+\frac{2}{3}g\right)(h+g)\left(f+\frac{5}{3}g\right)(h+2g)\left(f+\frac{8}{3}g\right)}\text{ etc.}$$

Furthermore, let it be

$$S=\int\frac{y^{m-1}dy}{(1-y^g)^{\frac{1}{3}}}\quad\text{and}\quad T=\int\frac{y^{n-1}dy}{(1-y^g)^{\frac{1}{3}}}$$

it will be

$$\frac{T}{S}=\frac{m\left(n+\frac{2}{3}g\right)(m+g)\left(n+\frac{5}{3}g\right)(m+2g)}{n\left(m+\frac{2}{3}g\right)(n+g)\left(m+\frac{5}{3}g\right)(n+2g)}\text{ etc.}$$

which two expression multiplied by each other give

$$\frac{RT}{QS}=\frac{fm\left(h+\frac{2}{3}g\right)\left(n+\frac{2}{3}g\right)(f+g)(m+g)\left(h+\frac{5}{3}g\right)\left(n+\frac{5}{3}g\right)}{hn\left(f+\frac{2}{3}g\right)\left(m+\frac{2}{3}g\right)(h+g)(n+g)\left(f+\frac{5}{3}g\right)\left(m+\frac{5}{3}g\right)}\text{ etc.}$$

§30 But this expression cannot be reduced to that one $\frac{27P}{8f^2gQ^3}$ was found to be equal to, if that one is not multiplied by $\frac{f}{f-\frac{1}{3}g}$, so that it is

$$\frac{27P}{8fg\left(f-\frac{1}{3}g\right)Q^3}=\frac{ff(f+g)(f+g)(f+g)(f+2g)}{\left(f-\frac{1}{3}g\right)\left(f+\frac{2}{3}g\right)\left(f+\frac{2}{3}g\right)\left(f+\frac{2}{3}g\right)\left(f+\frac{5}{3}g\right)\left(f+\frac{5}{3}g\right)}\text{ etc.;}$$

for, now the reduction is done by putting

$$m=f,\quad h=f-\frac{1}{3}g\quad\text{and}\quad n=f+\frac{1}{3}g.$$

Therefore, having substituted these values it will be

$$\frac{27P}{8fg\left(f-\frac{1}{3}g\right)Q^3}=\frac{RT}{QS}.$$

But since it is $S=Q$ and

$$R=\int\frac{y^{f-\frac{1}{3}g-1}dy}{(1-y^g)^{\frac{1}{3}}}=\frac{f+\frac{1}{3}g}{f-\frac{1}{3}g}\int\frac{y^{f+\frac{2}{3}g-1}dy}{(1-y^g)^{\frac{1}{3}}}$$

and

$$T=\int\frac{y^{f+\frac{1}{3}g-1}dy}{(1-y^g)^{\frac{1}{3}}},$$

one will obtain this equation by putting $y=z^3$

$$\int\frac{dz}{(1-z^3)^{\frac{1}{3}}}\cdot\int\frac{zdz}{(1-z^3)^{\frac{1}{3}}}=3fg(3f+g)\int\frac{z^{3f-1}dz}{(1-z^{3g})^{\frac{1}{3}}}\cdot\int\frac{z^{3f+g-1}dz}{(1-z^{3g})^{\frac{1}{3}}}\cdot\int\frac{z^{3f+2g-1}dz}{(1-z^{3g})^{\frac{1}{3}}}.$$

And if one puts $3f=a$, it will be

$$\int\frac{dz}{(1-z^3)^{\frac{1}{3}}}\cdot\int\frac{zdz}{(1-z^3)^{\frac{1}{3}}}=ag(a+g)\int\frac{z^{a-1}dz}{(1-z^{3g})^{\frac{1}{3}}}\cdot\int\frac{z^{a+g-1}dz}{(1-z^{3g})^{\frac{1}{3}}}\cdot\int\frac{z^{a+2g-1}dz}{(1-z^{3g})^{\frac{1}{3}}}.$$

§31 Let us put $p=1$ and $q=4$ and one will have

$$\frac{4^4P}{fg^3Q^4}=\frac{fff\,(f+g)\,(f+g)\,(f+g)}{\left(f+\frac{1}{4}g\right)\left(f+\frac{1}{4}g\right)\left(f+\frac{1}{4}g\right)\left(f+\frac{1}{4}g\right)\left(f+\frac{5}{4}g\right)\left(f+\frac{5}{4}g\right)}\text{ etc.}$$

and

$$\frac{R}{Q}=\frac{f\left(h+\frac{1}{4}g\right)(f+g)\left(h+\frac{5}{4}g\right)(f+2g)}{h\left(f+\frac{1}{4}g\right)(h+g)\left(f+\frac{5}{4}g\right)(h+2g)}\text{ etc.}$$

But let, as before, be

$$S=\int\frac{y^{m-1}dy}{(1-y^g)^{\frac{q-p}{q}}},\quad T=\int\frac{y^{n-1}dy}{(1-y^g)^{\frac{q-p}{q}}};$$

it will be

$$\frac{RST}{Q^3}=\frac{fff\left(h+\frac{1}{4}g\right)\left(m+\frac{1}{4}g\right)\left(n+\frac{1}{4}g\right)(f+g)}{hmn\left(f+\frac{1}{4}g\right)\left(f+\frac{1}{4}g\right)\left(f+\frac{1}{4}g\right)(h+g)}\text{etc.};$$

6 factors of this expression are to be transformed into four of that one, what will happen by putting

$$h = f + \frac{1}{4}\,, \quad m = f + \frac{2}{4}g \quad \text{and} \quad n = f + \frac{3}{4}g;$$

having done this one will have

$$4^4 P = fg^3 QRST.$$

Hence, because it is

$$P = \frac{1}{4}\int \frac{dz}{(1-z^4)^{\frac{3}{4}}} \cdot \int \frac{zdz}{(1-z^4)^{\frac{3}{4}}} \cdot \int \frac{zzdz}{(1-z^4)^{\frac{3}{4}}},$$

if on puts $y = z^4$ and $4f = a$, this equation will result

$$\int \frac{dz}{(1-z^4)^{\frac{3}{4}}} \cdot \int \frac{zdz}{(1-z^4)^{\frac{3}{4}}} \int \frac{zzdz}{(1-z^4)^{\frac{3}{4}}}$$

$$= ag^3 \int \frac{z^{a-1}dz}{(1-z^{4g})^{\frac{3}{4}}} \cdot \int \frac{z^{a+g-1}dz}{(1-z^{4g})^{\frac{3}{4}}} \cdot \int \frac{z^{a+2g-1}dz}{(1-z^{4g})^{\frac{3}{4}}} \cdot \int \frac{z^{a+3g-1}dz}{(1-z^{4g})^{\frac{3}{4}}},$$

whose connection to the preceding cases, in which it was $p = 1$, $q = 2$ and $p = 1$, $q = 3$, is easily seen.

§32 From these considerations it will therefore be possible to form all equations of this kind, which will result, if one puts $p = 1$ and $q =$ an arbitrary positive integer; of course, it will be

$$\text{I.}\quad \int \frac{dz}{\sqrt{1-z^2}}$$

$$= ag\int \frac{z^{a-1}dz}{\sqrt{1-z^{2g}}} \cdot \int \frac{z^{a+g-1}dz}{\sqrt{1-z^{2g}}}$$

$$\text{II.}\quad \int \frac{dz}{(1-z^3)^{\frac{2}{3}}} \cdot \int \frac{zdz}{(1-z^3)^{\frac{2}{3}}}$$

$$= ag^2\int \frac{z^{a-1}dz}{(1-z^{3g})^{\frac{2}{3}}} \cdot \int \frac{z^{a+g-1}dz}{(1-z^{3g})^{\frac{2}{3}}} \cdot \int \frac{z^{a+2g-1}dz}{(1-z^{3g})^{\frac{2}{3}}}$$

$$\text{III.}\quad \int \frac{dz}{(1-z^4)^{\frac{3}{4}}} \cdot \int \frac{zdz}{(1-z^4)^{\frac{3}{4}}} \cdot \int \frac{z^2dz}{(1-z^4)^{\frac{3}{4}}}$$

$$= ag^3\int \frac{z^{a-1}dz}{(1-z^{4g})^{\frac{3}{4}}} \cdot \int \frac{z^{a+g-1}dz}{(1-z^{4g})^{\frac{3}{4}}} \cdot \int \frac{z^{a+2g-1}dz}{(1-z^{4g})^{\frac{3}{4}}} \cdot \int \frac{z^{a+3g-1}dz}{(1-z^{4g})^{\frac{3}{4}}}$$

$$\text{IV.}\quad \int \frac{dz}{(1-z^5)^{\frac{4}{5}}} \cdot \int \frac{zdz}{(1-z^5)^{\frac{4}{5}}} \cdot \int \frac{z^2dz}{(1-z^5)^{\frac{4}{5}}} \cdot \int \frac{z^3dz}{(1-z^5)^{\frac{4}{5}}}$$

$$= ag^4\int \frac{z^{a-1}dz}{(1-z^{5g})^{\frac{4}{5}}} \cdot \int \frac{z^{a+g-1}dz}{(1-z^{5g})^{\frac{4}{5}}} \cdot \int \frac{z^{a+2g-1}dz}{(1-z^{5g})^{\frac{4}{5}}} \cdot \int \frac{z^{a+3g-1}dz}{(1-z^{5g})^{\frac{4}{5}}} \cdot \int \frac{z^{a+4g-1}dz}{(1-z^{5g})^{\frac{4}{5}}}$$

etc.

§33 But in order to also be able to calculate the equations, which result, if p is not $= 1$, let us put $p = 2$ and $q = 4$; having done this and while everything else remains the same as above it will be

$$\frac{4^4P}{3^4f^3gQ^4} = \frac{f\,(f+g)\,(f+g)\,(f+g)}{(f+\frac{3}{4}g)\,(f+\frac{3}{4}g)\,(f+\frac{3}{4}g)\,(f+\frac{3}{4}g)}\text{ etc.,}$$

where the remaining terms consisting of four factors are formed from these by augmenting the single factors by the quantity g. In like manner, it will on the other hand be

$$\frac{RST}{Q^3} = \frac{fff\,(h+\frac{3}{4}g)\,(m+\frac{3}{4}g)\,(n+\frac{3}{4}g)}{hmn\,(f+\frac{3}{4}g)\,(f+\frac{3}{4}g)\,(f+\frac{3}{4}g)}\text{ etc.,}$$

where each six factors constitute one revolution or period. But to make the comparison it is necessary that both series are contemplated this way

$$\frac{4^4 P}{3^4 f^2 g\,(f-\frac{1}{4}g)\,Q^4} = \frac{ff\,(f+g)\,(f+g)}{(f-\frac{1}{4}g)\,(f+\frac{3}{4}g)\,(f+\frac{3}{4}g)\,(f+\frac{3}{4}g)}\ \text{etc.}$$

$$\frac{hRST}{fQ^3} = \frac{ff\,(h+\frac{3}{4}g)\,(m+\frac{3}{4}g)\,(n+\frac{3}{4}g)\,(f+g)}{mn\,(f+\frac{3}{4}g)\,(f+\frac{3}{4}g)\,(f+\frac{3}{4}g)\,(h+g)}\ \text{etc.,}$$

of which this first is transformed into the second, so that it is

$$\frac{4^4 P}{3^4 fgh\,(f-\frac{1}{4}g)\,Q^4} = QRST,$$

if it is

$$h = f + \frac{1}{4}g\,, \quad m = f - \frac{1}{4}g \quad \text{and} \quad n = f + \frac{2}{4}g.$$

§34 Therefore, because it is

$$\begin{aligned} P &= \frac{3^4}{2}\int \frac{z^2dz}{(1-z^4)^{\frac{1}{4}}}\cdot\int \frac{z^5dz}{(1-z^4)^{\frac{1}{4}}}\int \frac{z^8dz}{(1-z^4)^{\frac{1}{4}}} \\ &= \frac{3^4}{32}\int \frac{dz}{(1-z^4)^{\frac{1}{4}}}\cdot\int \frac{zdz}{(1-z^4)^{\frac{1}{4}}}\cdot\int \frac{zzdz}{(1-z^4)^{\frac{1}{4}}} \end{aligned}$$

and

$$Q = \int \frac{y^{f-1}dy}{(1-y^g)^{\frac{1}{4}}}\,, \quad R = \int \frac{y^{f+\frac{1}{4}g-1}dy}{(1-y^g)^{\frac{1}{4}}}$$

$$S = \int \frac{y^{f-\frac{1}{4}g-1}dy}{(1-y^g)^{\frac{1}{4}}} = \frac{f+\frac{2}{4}g}{f-\frac{1}{4}g}\int \frac{y^{f+\frac{3}{4}g-1}dy}{(1-y^g)^{\frac{1}{4}}}$$

and

$$T = \int \frac{y^{f+\frac{2}{4}g-1}dy}{(1-y^g)^{\frac{1}{4}}},$$

from these having put $y = z^4$ and $4f = a$ the following equation is set up

$$\begin{aligned} &\int \frac{dz}{(1-z^4)^{\frac{1}{4}}}\cdot\int \frac{zdz}{(1-z^4)^{\frac{1}{4}}}\cdot\int \frac{zzdz}{(1-z^4)^{\frac{1}{4}}} \\ = ag\frac{(a+g)(a+2g)}{1\cdot 2}&\int \frac{z^{a-1}dz}{(1-z^{4g})^{\frac{1}{4}}}\cdot\int \frac{z^{a+g-1}dz}{(1-z^{4g})^{\frac{1}{4}}}\int \frac{z^{a+2g-1}dz}{(1-z^{4g})^{\frac{1}{4}}}\cdot\int \frac{z^{a+3g-1}dz}{(1-z^{4g})^{\frac{1}{4}}} \end{aligned}$$

§35 By proceeding this way one will find all following equations, whenever p is not $= 1$; and, if $p = 2$, it will be found

$$\text{I.}\quad \int \frac{dz}{(1-z^3)^{\frac{1}{3}}} \cdot \int \frac{zdz}{(1-z^3)^{\frac{1}{3}}} = ag(a+g) \int \frac{z^{a-1}dz}{(1-z^{3g})^{\frac{1}{3}}} \cdot \int \frac{z^{a+g-1}dz}{(1-z^{3g})^{\frac{1}{3}}} \cdot \int \frac{z^{a+2g-1}dz}{(1-z^{3g})^{\frac{1}{3}}}$$

$$\text{II.} \int \frac{dz}{(1-z^4)^{\frac{2}{4}}} \cdot \int \frac{zdz}{(1-z^4)^{\frac{2}{4}}} \cdot \int \frac{zzdz}{(1-z^4)^{\frac{2}{4}}}$$
$$= ag^2(a+g) \int \frac{z^{a-1}dz}{(1-z^{4g})^{\frac{2}{4}}} \cdot \int \frac{z^{a+g-1}dz}{(1-z^{4g})^{\frac{2}{4}}} \cdot \int \frac{z^{a+2g-1}dz}{(1-z^{4g})^{\frac{2}{4}}} \cdot \int \frac{z^{a+3g-1}dz}{(1-z^{4g})^{\frac{2}{4}}}$$

But in general, whatever q is, if one puts

$$\frac{dz}{(1-z^q)^{\frac{q-2}{q}}} = Xdz \quad \text{and} \quad \frac{z^{a-1}dz}{(1-z^{qg})^{\frac{q-2}{q}}} = Ydz,$$

it will be

$$\int Xdz \cdot \int zXdz \cdot \int z^2Xdz \cdots \int z^{q-2}Xdz$$
$$= ag^{q-2}(a+g) \int Ydz \cdot \int z^gYdz \cdot \int z^{2g}Ydz \cdots \int z^{(q-1)g}Ydz.$$

§36 In like manner, if it is $p = 3$ and one puts

$$\frac{dz}{(1-z^q)^{\frac{q-3}{q}}} = Xdz \quad \text{and} \quad \frac{z^{a-1}dz}{(1-z^{qg})^{\frac{q-3}{q}}} = Ydz$$

the following general equation will result

$$\int Xdz \cdot \int zXdz \cdot \int z^2Xdz \cdots \int z^{q-2}Xdz$$
$$= ag^{q-3}\frac{(a+g)(a+2g)}{1\cdot 2} \int Ydz \cdot \int z^gYdz \int z^{2g}Ydz \cdots \int z^{(q-1)g}Ydz.$$

And hence it is possible to collect all these formulas into one very general one. For, let p and q be arbitrary positive numbers and put

$$\frac{dz}{(1-z^q)^{\frac{q-p}{q}}} = Xdz \quad \text{and} \quad \frac{z^{a-1}dz}{(1-z^{qg})^{\frac{q-p}{q}}} = Ydz,$$

one will have

$$\int Xdz \cdot zXdz \cdot \int z^2 Xdz \cdots \int z^{q-2} Xdz$$
$$= ag^{q-p} \frac{(a+g)(a+2g)(a+3g)\cdots(a+(p-1)q)}{1\cdot 2\cdot 3\cdots(p-1)} \int Ydz \cdot \int z^g Ydz \cdot \int z^{2g} Ydz \cdots \int z^{(q-1)g} Ydz.$$

§37 But because it is

$$\int z^{q-1} Xdz = \frac{1}{p},$$

if both sides are multiplied by these factors, the following elegant equation will result

$$\frac{a(a+g)(a+2g)(a+3g)\cdots(a+(p-1)g)}{1\cdot 2\cdot 3\cdot 4\cdots p} g^{q-p}$$
$$= \frac{\int Xdz}{\int Ydz} \cdot \frac{\int zXdz}{\int z^g Ydz} \cdot \frac{\int z^2 Xdz}{\int z^{2g} Ydz} \cdot \frac{\int z^3 Xdz}{\int z^{3g} Ydz} \cdots \frac{\int z^{q-1} Xdz}{\int z^{(q-1)g} Ydz};$$

this expression contains all the ones found up to this point and is remarkable because of its extraordinary structure.

§38 Now I will proceed to another method, by means of which it is possible to get to expressions of this kind consisting of innumerable factors, which method is more accommodated to analysis. For, I observed that from the reduction of integral formulas to others one can obtain expressions of this kind. For, let this integral formula be propounded

$$\int x^{m-1} dx (1-x^{nq})^{\frac{p}{q}},$$

which is easily transformed into this expression

$$\frac{x^m (1-x^{nq})^{\frac{p+q}{q}}}{m} + \frac{m+(p+q)n}{m} \int x^{m+nq-1} dx (1-x^{nq})^{\frac{p}{q}}.$$

Therefore, if m and $\frac{p+q}{q}$ are positive numbers and the integrals are taken in such a way that they vanish for $x=0$, and then one puts $x=1$, it will be

$$\int x^{m-1} dx (1-x^{nq})^{\frac{p}{q}} = \frac{m+(p+q)n}{m} \int x^{n+nq-1} dx (1-x^{nq})^{\frac{p}{q}}.$$

§39 Further, since in like manner it is

$$\int x^{m+nq-1}dx(1-x^{nq})^{\frac{p}{q}} = \frac{m+(p+2q)n}{m+nq}\int x^{n+2nq-1}dx(1-x^{nq})^{\frac{p}{q}},$$

it will also be

$$\int x^{m-1}dx(1-x^{nq})^{\frac{p}{q}} = \frac{(m+(p+q)n)(m+(p+2q)n)}{m(m+nq)}\int x^{m+2nq-1}dx(1-x^{nq})^{\frac{p}{q}}.$$

Therefore, having continued this reduction to infinity it will be

$$\int x^{m-1}dx(1-x^{nq})^{\frac{p}{q}}$$

$$= \frac{(m+(p+q)n)(m+(p+2q)n)(m+(p+3q)n)\cdots(m+(p+\infty q)n)}{m(m+nq)(m+2nq)\cdots(m+\infty nq)}\int x^{m+\infty nq-1}dx(1-x^{nq})^{\frac{p}{q}}.$$

and in like manner it is

$$\int x^{\mu-1}dx(1-x^{nq})^{\frac{p}{q}}$$

$$= \frac{(\mu+(p+q)n)(\mu+(p+2q)n)(\mu+(p+3q)n)\cdots(\mu+(p+\infty q)n)}{\mu(\mu+nq)(\mu+2nq)\cdots(\mu+\infty nq)}\int x^{\mu+\infty nq-1}dx(1-x^{nq})^{\frac{p}{q}},$$

as long as m and μ and nq and $\frac{p+q}{q}$ are positive integer numbers or greater than zero.

§40 But since, if m is infinite, it is

$$\int x^{m}dx(1-x^{nq})^{\frac{p}{q}} = \int x^{m+\alpha}dx(1-x^{nq})^{\frac{p}{q}},$$

whatever finite number is assumed for α, as it was concluded in paragraph 38, it will also be

$$\int x^{m+\infty nq-1}dx(1-x^{nq})^{\frac{p}{q}} = \int x^{\mu+\infty nq-1}dx(1-x^{nq})^{\frac{p}{q}}.$$

Therefore, if the one of the preceding expressions is divided by the other one, this equation will result

$$\frac{\int x^{m-1}dx(1-x^{nq})^{\frac{p}{q}}}{\int x^{\mu-1}dx(1-x^{nq})^{\frac{p}{q}}}$$

$$= \frac{\mu(m+(p+q)n)(\mu+nq)(m+(p+2q)n)(\mu+2nq)(m+(p+3q)n)(\mu+3nq)}{m(\mu+(p+q)n)(m+nq)(\mu+(p+2q)n)(m+2nq)(\mu+(p+3q)n)(m+3nq)} \text{ etc. to infinity,}$$

by means of which expressions innumerable infinite products, whose values can be assigned by means of quadratures of curves, are found.

§41 If the one integral formula admits an integration, then one will have a nice infinite expression for the other integral formula. For, let it be $\mu = nq$; it will be

$$\int x^{\mu-1}dx(1-x^{nq})^{\frac{p}{q}} = \frac{1}{(p+q)q},$$

having substituted this value it will be

$$\int x^{m-1}dx(1-x^{nq})^{\frac{p}{q}} = \frac{1}{(p+q)n} \cdot \frac{nq(m+(p+q)n)2nq(m+(p+2q)n)3nq}{m(p+2q)n(m+nq)(p+3q)n(m+3nq)} \text{ etc.};$$

and using this infinite products can be found for innumerable integrals; at least in the case, in which $x = 1$, which is mainly desired in most cases, of course.

§42 Put n instead of nq and it will be

$$\int x^{m-1}dx(1-x^{n})^{\frac{p}{q}} = \frac{q}{(p+q)n} \cdot \frac{n(mq+(p+q)n)2n(mq+(p+2q)n)3n(mq+(p+3q)n)}{m(p+2q)n(m+n)(p+3q)n(m+2n)(p+4q)n} \text{ etc.},$$

which resolved into groups of two factors becomes simpler and it is

$$\int x^{m-1}dx(1-x^{n})^{\frac{p}{q}}$$
$$= \frac{q}{(p+q)n} \cdot \frac{1(mq+(p+q)n)}{m(p+2q)} \cdot \frac{2(mq+(p+2q)n)}{(m+n)(p+3q)} \cdot \frac{3(mq+(p+3q)n)}{(m+2n)(p+4q)} \cdot \text{etc.},$$

whence the following more notable examples are deduced:

$$\int \frac{dx}{\sqrt{1-xx}} = 1 \cdot \frac{1\cdot 4}{1\cdot 3}\cdot\frac{2\cdot 8}{3\cdot 5}\cdot\frac{3\cdot 12}{5\cdot 7}\cdot \text{ etc} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7}\text{ etc.,}$$

$$\int \frac{xdx}{\sqrt{1-xx}} = 1 \cdot \frac{1\cdot 6}{2\cdot 3}\cdot\frac{2\cdot 10}{4\cdot 5}\cdot\frac{3\cdot 14}{6\cdot 7}\cdot \text{ etc.} = 1,$$

$$\int \frac{x^2dx}{\sqrt{1-xx}} = 1 \cdot \frac{1\cdot 8}{3\cdot 3}\cdot\frac{2\cdot 12}{5\cdot 5}\cdot\frac{3\cdot 16}{7\cdot 7}\cdot \text{ etc} = \frac{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8}{3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7}\text{ etc.,}$$

$$\int \frac{dx}{\sqrt{1-x^3}} = \frac{2}{3}\cdot\frac{1\cdot 5\cdot 2\cdot 11\cdot 3\cdot 17\cdot 4\cdot 23\cdot 5\cdot 29}{1\cdot 3\cdot 4\cdot 5\cdot 7\cdot 7\cdot 10\cdot 9\cdot 13\cdot 11}\text{ etc.,}$$

$$\int \frac{xdx}{\sqrt{1-x^3}} = \frac{2}{3}\cdot\frac{1\cdot 7\cdot 2\cdot 13\cdot 3\cdot 19\cdot 4\cdot 25\cdot 5\cdot 31}{2\cdot 3\cdot 5\cdot 5\cdot 8\cdot 7\cdot 11\cdot 9\cdot 14\cdot 11}\text{ etc.,}$$

$$\int \frac{dx}{\sqrt{1-x^4}} = \frac{1}{2}\cdot\frac{1\cdot 6\cdot 2\cdot 14\cdot 3\cdot 22\cdot 4\cdot 30}{1\cdot 3\cdot 5\cdot 5\cdot 9\cdot 7\cdot 13\cdot 9}\text{ etc} = \frac{1}{2}\cdot\frac{2\cdot 3\cdot 4\cdot 7\cdot 6\cdot 11\cdot 8\cdot 15}{1\cdot 3\cdot 5\cdot 5\cdot 9\cdot 7\cdot 13\cdot 9}\text{ etc.,}$$

$$\int \frac{xxdx}{\sqrt{1-x^4}} = \frac{1}{2}\cdot\frac{1\cdot 10\cdot 2\cdot 18\cdot 3\cdot 26\cdot 4\cdot 34}{3\cdot 3\cdot 7\cdot 5\cdot 11\cdot 7\cdot 15\cdot 9}\text{ etc.,}$$

$$\int \frac{dx}{\sqrt[3]{1-x^3}} = \frac{1}{2}\cdot\frac{3\cdot 3\cdot 6\cdot 6\cdot 9\cdot 9\cdot 12\cdot 12}{1\cdot 5\cdot 4\cdot 8\cdot 7\cdot 11\cdot 10\cdot 14}\text{ etc.,}$$

$$\int \frac{dx}{\sqrt[4]{1-x^4}} = \frac{1}{3}\cdot\frac{4\cdot 4\cdot 8\cdot 8\cdot 12\cdot 12\cdot 16\cdot 16}{1\cdot 7\cdot 5\cdot 11\cdot 9\cdot 15\cdot 13\cdot 19}\text{ etc.}$$

Furthermore, these expressions deserve it to be noted

$$\int x^{m-1}dx(1-x^n)^{-\frac{m}{n}} = \frac{1}{n-m}\cdot\frac{n\cdot n\cdot 2n\cdot 2n\cdot 3n\cdot 3n}{m(2n-m)(m+n)(2n-m)(m+2n)(4n-m)}\text{ etc.,}$$

$$\int x^{m-1}dx(1-x^n)^{\frac{m-n}{n}}$$

$$= \frac{1}{m}\cdot\frac{n\cdot 2m\cdot 2n(2m+n)3n(2m+2n)4n(2m+3n)}{m(m+n)(m+n)(m+2n)(m+2n)(m+3n)(m+3n)(m+4n)}\text{ etc.}$$

§43 But because in the same way it is

$$\int x^{\mu-1}dx(1-x^{\vartheta})^{\frac{r}{s}} = \frac{s}{(r+s)\vartheta}\cdot\frac{1(\mu s+(r+s)\vartheta)2(\mu s+(r+2s)\vartheta)3(\mu s+(r+3s)\vartheta)}{\mu(r+2s)(\mu+\vartheta)(r+3s)(\mu+2\vartheta)(r+4s)}\text{ etc.,}$$

by dividing the first expression by this one it will be

$$\frac{\int x^{m-1}dx(1-x^n)^{\frac{p}{q}}}{\int x^{\mu-1}dx(1-x^{\vartheta})^{\frac{r}{s}}}$$
$$=\frac{(r+s)q\vartheta}{(p+q)sn}\cdot\frac{\mu(r+2s)(mq+(p+q)n)}{m(p+2q)(\mu s+(r+s)\vartheta)}\cdot\frac{(\mu+\vartheta)(r+3s)(mq+(p+2q)n)}{(m+n)(p+3q)(\mu s+(r+2s)\vartheta)}\cdot \text{ etc.}$$

Therefore, as often as this infinite expression has a finite value, so often the summation of the one integral formula can be reduced to the other one. But cases of this kind occur, whenever the factors of the numerator cancel the factors of the denominator, so that after the cancellation a finite number of factors remains. For, completely all reductions of integral formulas to other ones are contained in this expression.

§44 But in order to be able to compare more expressions of this kind to each other, it is advisable to assume them this way

$$\frac{\int x^{a-1}dx(1-x^b)^c}{\int x^{f-1}dx(1-x^g)^h}=\frac{(h+1)g}{(c+1)b}\cdot\frac{f(h+2)(a+(c+1)b)}{a(c+2)(f+(h+1)g)}\cdot\frac{(f+g)(h+3)(a+(c+2)b)}{(a+b)(c+3)(f+(h+2)g)}\cdot \text{ etc.,}$$

In like manner, it will be

$$\frac{\int x^{\alpha-1}dx(1-x^{\beta})^{\gamma}}{\int x^{\xi-1}dx(1-x^{\eta})^{\Theta}}=\frac{(\Theta+1)\eta}{(\gamma+1)\beta}\cdot\frac{\xi(\Theta+2)(\alpha+(\gamma+1)\beta)}{\alpha(\gamma+2)(\xi+(\Theta+1)\eta)}\cdot\frac{(\xi+\eta)(\Theta+3)(\alpha+(\gamma+2)\beta)}{(\alpha+\beta)(\gamma+3)(\xi+(\Theta+2)\eta)}\cdot \text{ etc.,}$$

which expressions, even though they do not differ in principle, nevertheless, since they have a different form, can be compared to each other.

§45 But that we now from these expressions find the same theorems we found above, let it be

$$\theta=\gamma=h=c, \qquad \eta=\beta=g=b;$$

it will be

$$\frac{\int x^{a-1}dx(1-x^b)^c}{\int x^{f-1}dx(1-x^b)^c}=\frac{f(a+c+1)b)(f+b)(a+(c+2)b)(f+2b)(a+(c+3b))}{a(f+c+1)b)(a+b)(f+(c+2)b)(a+2b)(f+(c+3b))}\text{ etc.}$$

and the other form

$$\frac{\int x^{\alpha-1}dx(1-x^b)^c}{\int x^{\zeta-1}dx(1-x^b)^c} = \frac{\zeta(\alpha+c+1)b)(\zeta+b)(\alpha+(c+2)b)(\zeta+2b)(\alpha+(c+3b))}{\alpha(\zeta+c+1)b)(\alpha+b)(\zeta+(c+2)b)(\alpha+2b)(\zeta+(c+3b))} \text{ etc.}$$

If the product of these expressions is put $= \frac{f}{a}$, it must be

$$\frac{(a+(c+1)b)(f+b)\xi(a+(c+1)b)}{(f+(c+1)b)(a+b)\alpha(\xi+(c+1)b)} = 1;$$

for, if this was the case, the product of the whole infinite expressions will become $= \frac{f}{a}$. But this will be achieved by putting

$$\alpha = a+(c+1)b\ , \quad \xi = f+(c+1)b$$

and it will be

$$c = -\frac{1}{2},$$

such that it is

$$\alpha = a+\frac{1}{2}b\ , \quad \xi = f+\frac{1}{2}b,$$

and it will hence be

$$\int \frac{x^{a-1}dx}{\sqrt{1-x^b}} \cdot \int \frac{x^{a+\frac{1}{2}b-1}dx}{\sqrt{1-x^b}} = \frac{f}{a}\int \frac{x^{f-1}dx}{\sqrt{1-x^b}} \cdot \int \frac{x^{a\frac{1}{2}b-1}dx}{\sqrt{1-x^b}};$$

or, if one puts $x = z^2$, it will be

$$\int \frac{z^{a-1}dz}{\sqrt{1-z^{2b}}} \cdot \int \frac{z^{a+b-1}dz}{\sqrt{1-z^{2b}}} = \frac{f}{a}\int \frac{z^{f-1}dz}{\sqrt{1-z^{2b}}} \cdot \int \frac{z^{f+b-1}dz}{\sqrt{1-z^{2b}}}$$

having put a and f instead of $2a$ and $2f$. But this equation is nothing else but the theorem found above in § 12; for, having put $f = b$ it is

$$\int \frac{z^{2b-1}dz}{\sqrt{1-z^{2b}}} = \frac{1}{b} \quad \text{and} \quad \int \frac{z^{b-1}dz}{\sqrt{1-z^{2b}}} = \frac{\pi}{2b},$$

whence it will be

$$\pi = 2ab\int \frac{z^{a-1}dz}{\sqrt{1-z^{2b}}} \cdot \int \frac{z^{a+b-1}dz}{\sqrt{1-z^{2b}}}.$$

§46 In like manner other theorems of this kind can be found; for, let it be

$$g = b\,, \quad h = c\,, \quad \eta = \beta = b \quad \text{and} \quad \Theta = \gamma$$

and let the case be in question, in which the product of the two expressions becomes $= 1$. But this will be obtained, if it is

$$\frac{f(a+(c+1)b)\xi(\alpha+(\gamma+1)b)}{a(f+(c+1)b)\alpha(\xi+(\gamma+1)b)} = 1,$$

what will happen by taking

$$\alpha = a + (c+1)b\,, \quad f = a + (\gamma+1)b\,, \quad \xi = a.$$

Therefore, having substituted these values the following rather elegant theorem will result

$$\frac{\int x^{a-1}dx(1-x^b)^c}{\int x^{a-1}dx(1-x^b)^\gamma} \cdot \frac{\int x^{a+(c+1)b-1}dx(1-x^b)^\gamma}{\int x^{a+(\gamma+1)b-1}dx(1-x^b)^c} = 1;$$

or, if one puts

$$c+1 = m \quad \text{and} \quad \gamma+1 = n,$$

one will have

$$\int \frac{x^{a-1}dx}{(1-x^b)^{1-m}} \cdot \int \frac{x^{a+mb-1}dx}{(1-x^b)^{1-n}} = \int \frac{x^{a-1}dx}{(1-x^b)^{1-n}} \cdot \int \frac{x^{a+nb-1}dx}{(1-x^b)^{1-m}}.$$

§47 Additionally, in another way one can find a nice theorem by putting $\gamma = h$ and $\theta = c$ while it remains $\eta = \beta = g = h$ and by causing that the product of the integral expressions becomes $= \frac{f}{a}$; for this to happen, it must necessarily be

$$\frac{(f+(c+1)b)(f+b)\xi(\alpha+(h+1)b)}{(f+(h+1)b)(a+b)\alpha(\xi+(c+1)b)} = 1.$$

But this will be achieved by taking

$$\alpha = a + (c+1)b\,, \quad \xi = f + (h+1)b,$$

whence one will find

$$c + h + 1 = 0 \quad \text{or} \quad h = -1 - c;$$

hence take

$$c = -\frac{1}{2} + n \quad \text{and} \quad h = -\frac{1}{2} - n,$$

and the following theorem will result

$$\frac{f}{a} = \frac{\int x^{a-1}dx(1-x^b)^{-\frac{1}{2}+n} \cdot \int x^{a+\left(\frac{1}{2}+n\right)b-1}dx(1-x^b)^{-\frac{1}{2}-n}}{\int x^{f-1}dx(1-x^b)^{-\frac{1}{2}-n} \cdot \int x^{f+\left(\frac{1}{2}-n\right)b-1}dx(1-x^b)^{-\frac{1}{2}+n}}$$

§48 Now let all exponents c, h, γ and θ be different, but $g = \beta = \eta = b$, and let the cases be in question, in which the product of both expressions becomes $= \frac{(h+1)(\theta+1)}{(c+1)(\gamma+1)}$. But this will happen, if it is

$$\frac{f(bh+2b)(a+(c+1)b)\xi(b\Theta+2b)(\alpha+(\gamma+1)b)}{a(bc+2b)(f+(h+1)b)\alpha(b\gamma+2b)(\xi+(\Theta+1)b)} = 1,$$

which factors I expressed in such a way, that they grow by the quantity b in the following terms. Now put

$$\xi + (\Theta+1)b = bh + 2b \quad \text{or} \quad \xi = b(1+h-\Theta)$$

and

$$\alpha + (\gamma+1)b = bc + 2b \quad \text{or} \quad \alpha = b(1+c-\gamma).$$

Further, let

$$f + (h+1)b = b\Theta + 2b \quad \text{or} \quad f = b(1+\Theta-h)$$

and

$$a + (c+1)b = b\gamma + 2b \quad \text{or} \quad a = b(1+\gamma-c).$$

Finally, it must be $\alpha = f$ and $\zeta = a$, which two equations require that it is

$$c - \gamma = \Theta - h \quad \text{or} \quad c + h = \gamma + \Theta.$$

Hence the following theorem will result

$$\frac{(h+1)(\Theta+1)}{(c+1)(\gamma+1)} = \frac{\int x^{b(1+\gamma-c)-1}dx(1-x^b)^c \cdot \int x^{b(1+c-\gamma)-1}dx(1-x^b)^\gamma}{\int x^{b(1+\theta-h)-1}dx(1-x^b)^h \cdot \int x^{b(1+h+\Theta)-1}dx(1-x^b)^\Theta},$$

as long as it is $c + h = \gamma + \Theta$.

§49 But additionally the expression can be forced to be $= 1$ another way, by putting

$$\alpha = a + (c+1)b \quad \text{and} \quad \zeta = f + (h+1)b, \quad f = b(\gamma+2), \quad a = b(\theta+2),$$

such that it is

$$\alpha = b(3 + c + \Theta) \quad \text{and} \quad \xi = b(3 + h + \gamma).$$

But further it must be

$$\xi + (\Theta + 1)b = bh + 2b \quad \text{and} \quad \alpha + (\gamma+1)b = bc + 2b;$$

this demands that it is

$$\gamma + \Theta + 2 = 0.$$

Therefore, put

$$\gamma = -1 + n \quad \text{and} \quad \Theta = -1 - n$$

But if it is required that the product of both expressions is $= \frac{f(h+1)(\theta+1)}{a(c+1)(\gamma+1)}$, this will be achieved by putting

$$\alpha = a + (c+1)b\ , \quad \xi = f + (h+1)b\ , \quad f = b(\gamma+1)\ , \quad a = b(\Theta+1),$$

whence it will be

$$\alpha = b(2 + c + \Theta) \quad \text{and} \quad \xi = b(2 + h + \gamma).$$

Finally, it must be

$$\gamma + \Theta + 1 = 0.$$

Put

$$\gamma = -\frac{1}{2} + n \quad \text{and} \quad \Theta = -\frac{1}{2} + n$$

and one will have this theorem

$$\frac{h+1}{c+1} = \frac{\int x^{b\left(\frac{1}{2}-n\right)-1}dx(1-x^b)^c}{\int x^{b\left(\frac{1}{2}+n\right)-1}dx(1-x^b)^h} \cdot \frac{\int x^{b\left(\frac{3}{2}+c-n\right)-1}dx(1-x^b)^{-\frac{1}{2}+n}}{\int x^{b\left(\frac{3}{2}+h+n\right)-1}dx(1-x^b)^{-\frac{1}{2}-n}};$$

here it is to be noted that the exponents c, h, $-\frac{1}{2}+n$, $-\frac{1}{2}-n$ can certainly be negative numbers, but of such a kind that they become positive if 1 is added to them; for, otherwise the integrals would not obtain a finite value in the case $x = 1$.

§50 Therefore, as I not only discovered the theorem found above on products of two integrals formulas by this more direct method, but also found new not less remarkable ones, so, if in like manner three expressions of this kind are multiplied by each other, many theorems on the products of three integral formulas will result and it will be possible to proceed further to an arbitrary number of factors; but since this investigation requires very cumbersome calculations, that even the letters of the alphabet hardly suffice, I will be contented both by the main theorems given and the way of proving them.